Dynamics of Liquid-Solid Fluidized Bed Expansion

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The dynamic response of bed height in liquid-solid fluidized beds to a change in fluidizing velocity was considered. A linearized model was developed such that the time constant for the system could be estimated with the parameters characterizing the system.

Spherical particles, with sizes ranging from 6 to 7 to 70 to 80 mesh and densities ranging from 1.08 g./cc. to 2.5 g./cc., were fluidized by water in a 2-in. diameter glass column. The experimental response to step and rectangular pulse inputs in fluidizing velocity were measured and frequency response data were also obtained.

Agreement between the experimental data and the model was quite reasonable in view of the known nonlinearity of fluidization systems. These results show that for control purposes the system behaves as a first order time constant process.

The use of fluidization in industry is very wide and increasing in scope. In many of the fluidization applications the maintenance of a given porosity or bed height is either desirable or essential. For instance the neutron density of a fluidized nuclear reactor will be a function of porosity, and failure to maintain a given porosity within certain limits will result in a noncritical condition. Also in any application of fluidization there will be a porosity determined by design procedures or by experimental work at which the process is to operate. Therefore the problem of process control is becoming increasingly apparent in the area of fluidization as in all phases of the chemical process industry (*1*).

The disturbances a process may experience are usually random in nature, and it is very difficult to determine its response to random inputs. However it is possible to describe the response of a linear system to any input as a combination of its responses to certain regular inputs. In particular the transient response to a step input and the steady state response to a sinusoidal input are useful.

The fluidized bed presents a typical problem of nonlinearity. The nonlinear element existing in fluidized bed analysis might be considered roughly analogous to the nonlinearities occurring in heat and mass transfer. Just as the heat and mass transfer coefficients are dependent upon the fluid flow conditions, the relationship between fluidizing velocity and porosity is also dependent on flow conditions. Although considerable work has been done on the analysis of the dynamic behavior of heat and mass transfer equipment, almost none has been done on the fluidized bed.

There are two articles (*2, 3*) reporting the dynamic behavior of the liquid-solid fluidized bed expansion. Both are based on nonlinear models, and one article (*2*) does not contain any experimental data.

The major purpose of this work was to develop a linearized model to describe the dynamic behavior of the liquid-solid fluidized bed and to verify that model with experimental results. It was hoped that a sufficiently accurate representation could be obtained so as to be useful in the design of control systems.

DEVELOPMENT OF LINEARIZED MATHEMATICAL MODEL

Richardson and Zaki (*4*) showed that for steady state behavior the porosity can be related to the fluidizing velocity by

$$\phi_{ss} = U_s \epsilon^n_{ss} \quad (1)$$

In considering the response of a liquid-solid fluidized bed to a step input in fluidizing velocity Slis, Willemse, and Kramers (*3*) extended Equation (1) to the unsteady state case and obtained the expression for the rate of change of the bed height as

$$\frac{dh}{dt} = \phi_1 - U_s \epsilon^n_h \quad (2)$$

For the steady state case $dh/dt = 0$, $\phi_1 = \phi_{ss}$, and Equation (2) reduces to Equation (1). Equation (2) can be linearized by letting

$$\phi_1 = \phi_{ss} + \Delta\phi \quad (3)$$

$$h = h_{ss} + \Delta h \quad (4)$$

$$\epsilon_h = \epsilon_{ss} + \Delta\epsilon \quad (5)$$

Now ϵ^n_h can be expanded in a Taylor series about ϵ_{ss} to give

$$\epsilon^n_h = \epsilon^n_{ss} + \epsilon^{n-1}_{ss}\,\Delta\epsilon + \ldots \quad (6)$$

Substitution of Equations (3), (4), and (6) into Equation (2) gives

$$\frac{dh_{ss}}{dt} + \frac{d(h)}{dt} = \phi_{ss} + \Delta\phi - U_s\,\epsilon^n_{ss} + n\,\epsilon^{n-1}_{ss}\,\Delta\epsilon \quad (7)$$

Since the amount of solids in the bed remains constant, an instantaneous material balance is written as

$$(1-\epsilon)\,h = (1-\epsilon_{ss})\,h_{ss} \quad (8)$$

Differentiation of Equation (8) gives

$$\frac{d\epsilon}{dh} = \frac{(1-\epsilon_{ss})\,h_{ss}}{h^2} \quad (9)$$

For small changes in h Equation (9) becomes

$$\Delta\epsilon = \left(\frac{1-\epsilon_{ss}}{h_{ss}}\right)\Delta h \quad (10)$$

Substituting Equation (10) into Equation (7) one gets

$$\frac{d(h_{ss})}{dt} + \frac{d(\Delta h)}{dt} = \phi_{ss} + \Delta\phi - U_s\,\epsilon^n_{ss} - U_s\,n\,\epsilon^{n-1}_{ss}\left(\frac{1-\epsilon_{ss}}{h_{ss}}\right)\Delta h$$

but

$$\frac{dh_{ss}}{dt} = 0 = \phi_{ss} - U_s\,\epsilon^n_{ss} \quad (11)$$

and consequently Equation (11) reduces to

$$\frac{d(\Delta h)}{dt} + \frac{1}{\dfrac{h_{ss}}{U_s\,n\epsilon^{n-1}_{ss}\,(1-\epsilon_{ss})}}\,\Delta h = \Delta\phi \quad (12)$$

$\Delta\phi$ can be arbitrarily written as

$$\frac{1}{\dfrac{h_{ss}}{U_s\,n\epsilon^{n-1}_{ss}\,(1-\epsilon_{ss})}}\,A_i = \Delta\phi \quad (13)$$

Substitution of Equation (13) into Equation (12) and a change of dependent variable from Δh to h gives

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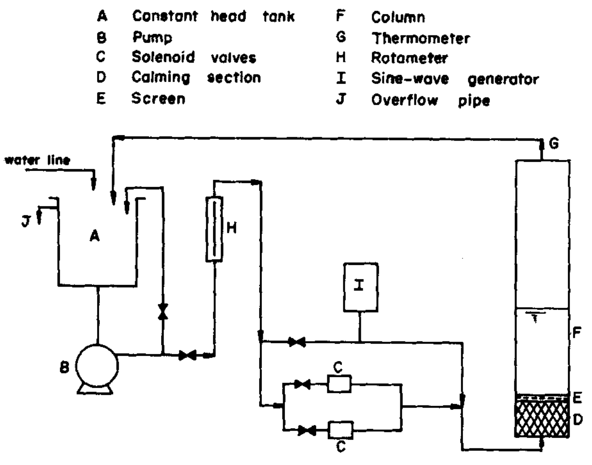

Fig. 1. Diagram of experimental apparatus.

TABLE 1. PROPERTIES OF MATERIALS

Group	Size	Material
I	6-7 U.S. mesh	Borosilicate glass beads
II	25-30 U.S. mesh	Nalcite ion exchange resin
III	40-45 U.S. mesh	M.M.M. glass beads
IV	60-70 U.S. mesh	M.M.M. glass beads
V	70-80 U.S. mesh	M.M.M. glass beads

Group	Diameter, (cm.)	Density, (g./cm.)	U_s (cal.), (cm./sec.)	U_s (exp.), (cm./sec.)
I	0.283-0.336	2.5	35.8	32.1
II	0.059-0.071	1.08	1.00	1.15
III	0.035-0.042	2.5	4.82	5.06
IV	0.021-0.025	2.5	2.68	2.83
V	0.0177-0.021	2.5	2.12	2.35

$$\frac{dh}{dt} + \frac{1}{\dfrac{h_{ss}}{U_s n\epsilon^{n-1}_{ss}(1-\epsilon_{ss})}} h = \frac{A_i}{\dfrac{h_{ss}}{U_s n\epsilon^{n-1}_{ss}(1-\epsilon_{ss})}} \tag{14}$$

If one considers the liquid-solid fluidized bed to be a first-order (time constant) system, the rate of change of height can be written as (5)

$$\frac{dh}{dt} + \frac{1}{T}h = \frac{h_i}{T} \tag{15}$$

A comparison of Equation (15) with Equation (14) shows that

$$\frac{dh}{dt} + \frac{1}{T}h = \frac{A_i}{T} \tag{16}$$

where the time constant T can be related to the parameters characterizing the fluidization system as

$$T = \frac{h_{ss}}{U_s n\epsilon^{n-1}_{ss}(1-\epsilon_{ss})} \tag{17}$$

By substituting Equation (1) into Equation (17) the time constant can also be written as

$$T = \frac{h_{ss}\epsilon_{ss}}{\phi_{ss} n(1-\epsilon_{ss})} \tag{18}$$

Either Equation (17) or Equation (18) can be used to calculate the theoretical time constant of the liquid-solid fluidized bed.

The solutions of Equation (15) or (16) for various input functions are given below. Integration of Equation (14) or Equation (16) gives, for $A_i = A =$ step size

$$-\frac{t}{T} = \ln\left[1 - \left(\frac{h - h_{ss}}{A}\right)\right] \tag{19}$$

as a solution for the step input. Equation (19) can also be put in the form

$$\frac{h - h_{ss}}{A} = 1 - e^{-t/T} \tag{20}$$

The input considered next is composed of two step inputs separated by a time interval a. The second step is of the same magnitude as the first but opposite in direction. The solution to Equation (15) for this input is

$$\frac{h - h_{ss}}{A} = (1 - e^{-t/T}) - u(t-a) + e^{-\frac{(t-a)}{T}} \quad (t-a) \geq 0 \tag{21}$$

$$u(t-a) = 0 \quad \text{for} \quad 0 < t < a$$
$$= 1 \quad \text{for} \quad a < t \tag{22}$$

For the input $h_i = h^* \sin \omega t$ the steady state solution to Equation (15) is

$$\frac{h - h_{ss}}{h^*} = \frac{\sin(\omega t + \alpha)}{\sqrt{1 + T^2\omega^2}} \tag{23}$$

From Equation (23) it is seen that the amplitude ratio is given by

$$\text{A. R.} = \frac{\text{output amplitude}}{\text{input amplitude}} = \frac{h^*/\sqrt{1+\omega^2T^2}}{h^*} = \frac{1}{\sqrt{1+\omega^2T^2}} \tag{24}$$

and the phase lag by

$$\alpha = -\arctan \omega T \tag{25}$$

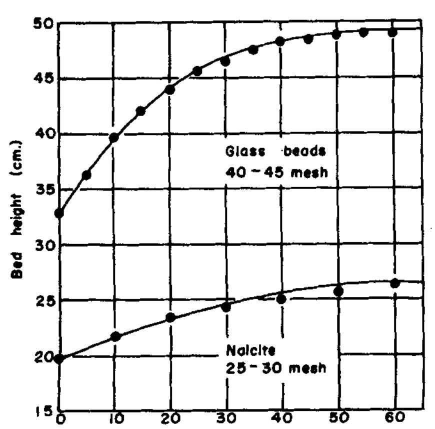

Fig. 2. Time vs. height for step input.

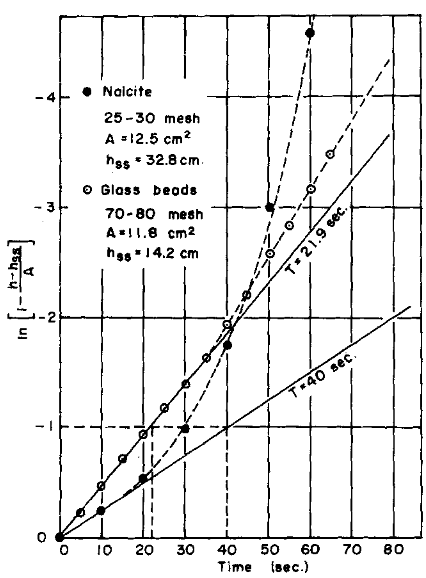

Fig. 3. Experimental determination of time constant.

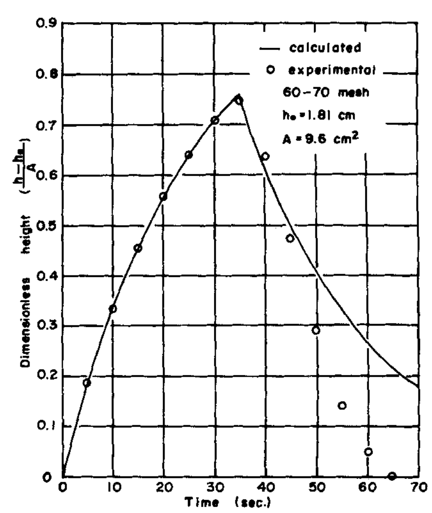

Fig. 4. Response of liquid-solid fluidized bed to pulse input.

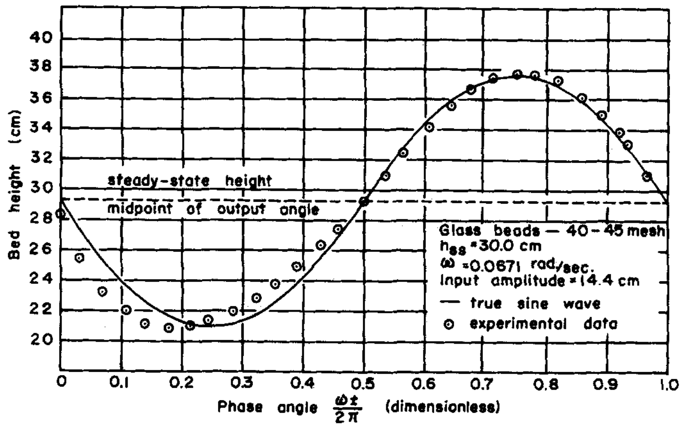

Fig. 5. Bed height vs. phase angle for sinusoidal input in fluidizing velocity.

EXPERIMENTAL

A schematic diagram of the experimental setup is shown in Figure 1.

The fluidization column used was a 24-in. high, 1.982-in. diameter glass column. When it was desired to produce a step or pulse input in flow rate, the flow was directed through a circuit consisting of two solenoid valves in parallel. Each of these valves was in series with a globe valve which could be used to regulate the flow through that particular line when it was open. The solenoids used were ⅜ in. I.D., two-way valves. They were connected through one electrical switch in such a manner that one would be open and the other closed on each switch position.

A constant angular velocity input motion which was supplied by a motorized variable speed transmission was converted to a linear sinusoidal motion by a scotch yoke mechanism. This sinusoidal displacement of a piston in a cylinder inserted in the steady flow line produced sinusoidal variations in the fluid velocity.

Five different groups of particles were employed in this experiment. The characteristics of the particles are listed in Table 1.

Steady state fluidization data were obtained first by setting the flow rate and other operating variables at constant values. A switch was then thrown to cause both solenoid valves to operate and effect a change in flow from one line to the other, giving a step change in flow rate to the bed. The instantaneous bed height was read at regular time intervals during the transition from the initial steady state height to the final steady state height. Figure 2 shows a typical step change response curve.

The data taken on the response to a step input in fluidizing velocity was to be used to verify the linearized model developed in the previous section. Referring to Equation (19) one can see that a plot of the group $\ln\left[1 - \frac{h - h_{ss}}{A}\right]$ vs. time t should yield a straight line of slope $-1/T$. Therefore by plotting the experimental data expressed in this form an experimental determination of the time constant T was obtained for each experimental run. A typical determination of experimental time constant is shown in Figure 3 for 70- to 80-mesh glass beads. Since the points for 70- to 80-mesh glass beads fall very nearly on a straight line for the first 35 sec., there is very little question as to how the straight line is to be drawn. However in cases where considerable curvature was evident, the straight line was always drawn so as to represent the tangent to the curve at the origin.

A case where this procedure was necessary is shown in Figure 3 for 25- to 30-mesh nalcite particles. The rapid deviation of this curve from the straight line is caused by the fact that the curve represents a response to a step-down in flow rate, and this case is not approximated nearly as well as the response to a step-up by the linearized model. It is seen however that the approximation is accurate for a short period of time.

In order to obtain some idea of the applicability of the linearized model it was necessary to devise an arbitrary test. A time interval $t_{0.02}$ was defined in such a manner that the expression

$$\left|\left(\frac{h - h_{ss}}{A}\right)_{exp} - \left(\frac{h - h_{ss}}{A}\right)_{cal}\right| < 0.02 \qquad (26)$$

was valid within the interval. The subscripts exp and cal in expression (26) refer to experimental values and values obtained by the linearized model respectively.

The response to a pulse composed of two step inputs was obtained in much the same manner as the response to a step change. The only difference in procedure was to effect a second step change in flow back to the original flow rate at some time before the bed height reached the steady state value corresponding to the intermediate flow rate. Figure 4 shows a typical pulse response curve.

The sine wave generator was adjusted for lever arm length and rotational velocity in order to obtain the desired input amplitude and frequency. The sine input was then superimposed on the steady-state flow rate. After one waited a few cycles to insure that the transient effects had sufficiently subsided, the maximum and minimum in bed height were read and recorded. Phase lag data were obtained by observing and recording the time difference between the reversal in piston direction and the reversal in bed direction. This was not possible when operating at high frequencies, where the phase lag was close to 90 deg. and the time interval between reversals was very small. The response to a sinusoidal input is shown in Figure 5. Figures 6 and 7 are typical frequency response curves.

When the 6- to 7-mesh glass beads were fluidized, a tendency toward aggregative fluidization was clearly observed. Two distinct phases were observed for aggregative fluidization. One consisted of areas densely populated with particles and the other of bubbles of water traveling up through the bed. The aggregative fluidization data had to be treated somewhat semiempirically.

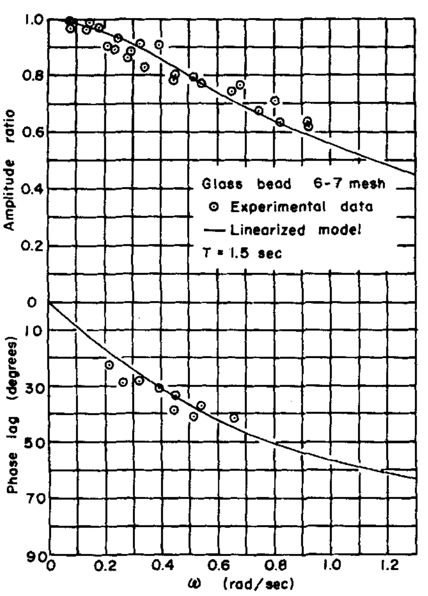

Fig. 6. Frequency response of liquid-solid fluidized bed for 6- to 7-mesh glass beads.

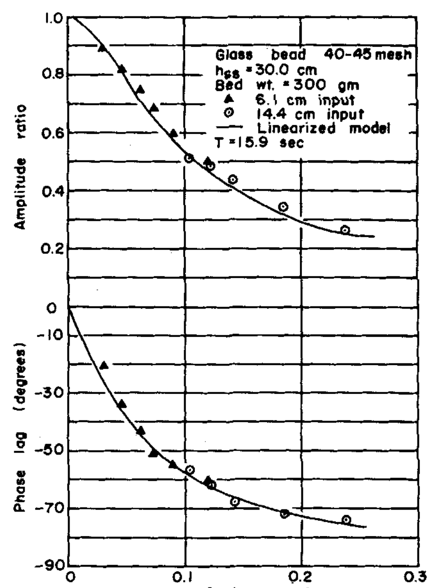

Fig. 7. Frequency response of liquid-solid fluidized bed for 40- to 45-mesh glass beads.

The output amplitude was determined as the difference between the maximum and minimum in bed height expansion divided by 2. The input amplitude was a sinusoidally varying flow rate. For use in computation this had to be converted into the same units as the output amplitude. This was accomplished by taking the difference between maximum and minimum bed height divided by 2 from the steady state curve relating bed height and flow rate for the system being studied.

The experimental phase lag was determined by obtaining the time interval between the reversal in piston motion and the reversal in bed height motion. The phase lag was then determined as

$$\tan(90° - \alpha) = \frac{4\omega(\Delta t)}{2\pi} \quad 0° \leq \alpha \leq 90° \tag{27}$$

This method was very limited since Δt is quite small at high frequencies and could not be measured by observation.

DISCUSSION OF RESULTS AND CONCLUSIONS

There is a factor which makes fluidization very difficult to analyze from a dynamic standpoint. This involves the mechanism of bed expansion and contraction. The fact that the mechanism for expansion is different from that of contraction presents the difficulty. A quantitative description of these mechanisms may be found in reference 3. It should suffice here to mention that there is an essential difference between expansion and contraction, and therefore the response to a decreasing flow rate would differ from that to an increasing flow rate. Because of this behavior it would be impossible to describe the dynamic behavior of a fluidized bed accurately by one model. It should however be possible to describe the dynamic behavior for small fluctuations by a single, linear model, and this is what has been attempted. The necessity for this type of representaion is obvious when it is considered that the fluctuations in flow rate, which might occur in an actual process, are random and small.

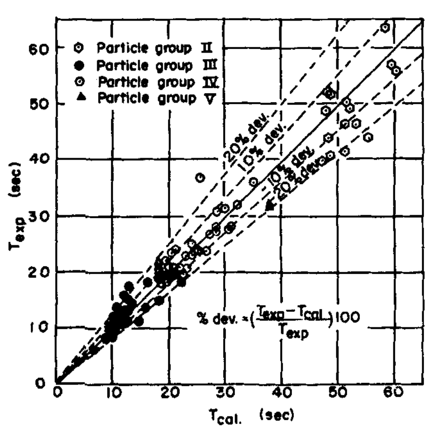

Fig. 8. Comparison of calculated and experimental time constants for response to step input.

For the response to a step input there are two types of deviations to be considered. The first involves the deviations of the time constants calculated from the linearized model from the experimental time constants as shown in Figure 8. Since no systematic variation in these deviations could be related to any of the operational variables, such as bed height, porosity, or size of step, it must be assumed that they are the result of inaccuracies in calculation and inaccuracy inherent in the graphical method of obtaining experimental time constants. On the other hand the time interval over which the experimental response follows the linearized model is definitely a function of the operating conditions. In general the response to a step-up is described accurately over a much greater time interval than the response to a step-down.

Slis, et al. (3) indicated that a step-down in fluidizing velocity results in a straight line response of bed height. This response can be represented by

$$h = (\phi_1 - \phi_{ss})\, t + h_{ss} \tag{28}$$

The linearized model however gives a response of

$$h = A\,(1 - e^{-t/T}) + h_{ss} \tag{29}$$

Rearranging Equations (28) and (29) and taking their difference one gets

$$\left(\frac{h - h_{ss}}{A}\right)_{\text{hyp}} - \left(\frac{h - h_{ss}}{A}\right)_{\text{cal}} = kt - (1 - e^{-t/T}) \tag{30}$$

where $k = (\phi_1 - \phi_{ss})/A$. Here the subscript (hyp) stands for hypothetical response based on Equation (28) and (cal) for the response calculated from the linearized model Equation (29). Comparing Equation (30) with Equation (26) one sees that

$$kt_{0.02} - (1 - e^{-t_{0.02}/T}) = 0.02 \tag{31}$$

This can be rewritten as

$$e^{-t_{0.02}/T} + kt_{0.02} = 1.02 \tag{32}$$

Equation (32) can be further simplified by noting that $\Delta\phi = A/T$ from Equation (13). Since $k = \Delta\phi/A$ one can write

$$k \cong \frac{A/T}{A} = \frac{1}{T} \tag{33}$$

and Equation (32) becomes

$$e^{-t_{0.02}/T} + \frac{t_{0.02}}{T} = 1.02 \tag{34}$$

There will be only one value of $(t_{0.02}/T)$ which will satisfy Equation (34). This was found to be $(t_{0.02}/T) =$ 0.205. The actual values of $(t_{0.02}/T)$ vary from 0.1 to 0.35 with a mean of 0.22. Part of this error is undoubtedly due to the trial-and-error method of obtaining the $t_{0.02}$ and the remainder to the inaccuracy of Equation (34). It should be emphasized that the value 0.02 was chosen arbitrarily and that any other desired value may be picked.

Figure 5 shows that a small amount of distortion in the output wave form was observed. This is reasonable and can be attributed chiefly to two sources. First the input in terms of flow rate is a true sine wave (subject to experimental error), but the input in terms of equivalent bed height is not. This is because of the nonlinear nature of the steady state relation between bed height and fluidizing velocity. Therefore, since the input is not exactly sinusoidal, one could not expect the output to be. This source of distortion is of course dependent on the input amplitude and becomes negligible as the input amplitude approaches zero. The second source of output distortion is the difference in mechanisms for expansion and contraction of the bed. This effect could be quite important and is not particularly related to the input amplitude. Therefore a small amount of distortion can be expected for any input amplitude. It should be noted that the response curve presented in Figure 5 is the result of rather extreme conditions. The input in that case is 14.4 cm. or approximately 0.5 h_{ss}. Also, by the manner of constructing that figure, all of the distortion has been concentrated in the lower half of the curve. The reason for this is that one experimental point had to be selected as a reference point for the remainder of the curve. This was chosen as the point where $(\omega t)/(2\pi)$ equals 0.75. In the actual response it is doubtful that all of the distortion would be concentrated in one area of the curve as it is shown. The authors feel however that owing to the mechanism for bed contraction (the bed tends to contract at a higher rate than it expands for the same input) that more distortion would be expected in the lower half of the cycle than in the upper half, and Figure 5 was constructed to emphasize this.

A limited number of data were obtained on the phase lag relationship. The data presented were obtained by averaging the phase lag at the top of the cycle with that at the bottom. This procedure generally gave results in fairly good agreement with the linearized model, as shown in Figure 5. The agreement of the data shown with the curve calculated with Equation (23) is very good.

A final point on the deviations encountered in the response to a sinusoidal input is concerned with the shift in the steady state height h_{ss} which was observed. This can be seen in Figure 5. This was most likely due to the tendency for the bed to contract more rapidly than expand as mentioned previously.

Slis, et al. (3) reported an experimental response to a step-up in flow which was linear for the first part of the response. It turned out in fact that the time interval over which this linear response was observed was defined by

$$0 < t < t^*$$

where

$$t^* = \frac{\epsilon_{ss} h_{ss}}{n \phi_{ss} (1 - \epsilon_{ss})} \qquad (35)$$

It should be noted that this equation for t^* is the same equation used to calculate T. In the present experimental work the time constant T was usually in the neighborhood of 10 to 20 sec. It should have been possible to detect this linear relationship quite easily, but it was not observed. The response to a step-down was quite definitely linear as predicted in reference 3. This agreement in one case and disagreement in another related case is very interesting. A possible explanation involved the effect of particle size distribution on the response of fluidizing beds to upsets. If all of the particles are of uniform density, as is usually the case, the smaller particles will tend to concentrate at the top of the bed and the larger particles at the bottom. For particles of uniform density the terminal falling velocity will increase with increasing diameter. This will also cause an increase in Reynolds number and a corresponding decrease in the empirical constant n. However the increase in U_s will always be proportionally greater than the decrease in n. Referring to Equation (17) one sees that the time constant will decrease with increasing particle diameter owing to the effect described above, and larger particles can be expected to show a faster response to upsets than small particles. Now if the bed is at steady state conditions and is subject to a step-up in fluidizing velocity, the particles at the top of the bed, the smaller particles, will respond most slowly with an increase in response speed as one goes toward the bottom of the bed. The initial response of the bed height will be due almost entirely to these slow responding particles at the top of the bed, but there will be considerable particle mixing and interaction within the bed; as a result the particle distribution at the top of the bed will be continually changing and yielding different response characteristics. It must be pointed out that this discussion pertains to the condition at the top of the bed at times before t^*. After t^* which denotes the time at which a disturbance is to reach the top of the bed there will definitely be a change in porosity at this point, and this will cause varying response characteristics. When one considers the response to a step-down, the situation is reversed. The particles at the bottom will still respond faster, but the direction of this response is away from the top of the bed. The particles within the bed will not be affected greatly by particles below them, and the entire bed will settle much as in sedimentation.

The smaller the particle size range the more negligible the effects described above would be. It is supposed that the particles used by Slis, et al. (3) were very uniform. It is important to be conscious of the phenomena described however, since in most applications there is a quite sizable particle size range.

Most of the fluidization processes fall into the category of aggregative fluidization. This is particularly true of gas-solid fluidization, and several sources have recently reported having observed it in liquid-solid systems (6). In liquid-solid fluidization aggregative fluidization occurs when large, heavy particles are fluidized with a relatively nonviscous liquid such as water.

A problem was encountered in trying to apply the linearized model to aggregative fluidization represented by the 6- to 7-mesh glass beads. The terminal falling velocity of the particles was determined and n calculated by the appropriate correlation. These values were 35.8 cm./sec. and 2.39 respectively. For particulate fluidization therefore one would expect a steady state relationship from Richardson and Zaki (4) of the form

$$\phi_{ss} = U_s \epsilon^n_{ss} = 35.8 \epsilon^{2.39}_{ss} \qquad (36)$$

For aggregate fluidization this is no longer valid. From steady state data obtained for this system it was found experimentally that the relationship

$$\phi_{ss} = a \epsilon^b_{ss} \qquad (37)$$

could be used to represent the data. For this case a was found to be 23.8 and b was determined as 2.5. It was found however that the use of the values of U_s and n in determining the system time constant gave satisfactory results, even though these were not the same as the experimental parameters for the steady state relationship. This value of T was 1.5 sec. It can be seen from Figure 6 that use of this time constant gave satisfactory agreement with the experimental data. This indicates that for liquid-solid agggregative fluidized beds the dilute phase does not contribute significantly to the time constant of the system. The value of T based on a and b in Equation (37) was 2 sec.

NOTATION

- A = maximum bed expansion (step height), L
- a = constant in Equation (37), L/θ
- B = slope of linear input of height variable, L/θ
- b = constant in Equation (37), dimensionless
- D_p = diameter of particle, L
- h = bed height at time, t, L
- h_{ss} = steady state bed height, L
- h^* = input amplitude in bed height for frequency response, L
- k = constant in Equation (35), $1/\theta$
- n = constant in Equation (1), dimensionless
- N_{Re} = Reynolds number, dimensionless
- t = time, θ
- $t_{0.02}$ = time interval defined by Equation (26), θ
- T = theoretical time constant, θ
- U_s = terminal falling velocity of particles, L/θ

Greek Letters

- α = phase lag, deg.
- ω = frequency, $1/\theta$
- ϵ = porosity dimensionless
- ϵ_{ss} = steady state porosity, dimensionless
- ϵ_h = porosity at top of bed, dimensionless
- ϕ = superficial liquid velocity, L/θ
- ϕ_{ss} = steady state superficial liquid velocity, L/θ
- ϕ_1 = steady state superficial liquid velocity after step input, L/θ

LITERATURE CITED

1. Williams, Th. J., "Systems Engineering for the Process Industries," McGraw-Hill, New York (1961).
2. Martin Nuclear Company, *Technical Report MND-FBR-1696* (1959).
3. Slis, P. L., T. H. Willemse, and H. Kramers, *Appl. Sci. Res.*, **8A**, 209-218 (1959).
4. Richardson, J. F., and W. N. Zaki, *Trans. Inst. Chem. Engrs.*, **32**, 34 (1954).
5. Ceaglske, N. H., "Automatic Process Control for Chemical Engineers," Wiley, New York (1956).
6. Kelly, V. P., *U.S. Government Report HW-54991.*

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